Robust synchronization of chaotic systems

Ljupco Kocarev
Institute for Nonlinear Science, University of California, San Diego, 9500 Gilman Drive, La Jolla, California 92093-0402 and Department of Physics, College of William and Mary, Williamsburg, Virginia 23187-8795

Ulrich Parlitz
Drittes Physikalisches Institut, Universität Göttingen, Bürgerstraße 42-44, D-37073 Göttingen, Germany

Reggie Brown
Department of Physics and Department of Applied Science, College of William and Mary, Williamsburg, Virginia 23187-8795

(Received 6 October 1999)

The question of robustness of synchronization with respect to small arbitrary perturbations of the underlying dynamical systems is addressed. We present examples of chaos synchronization demonstrating that normal hyperbolicity is a necessary and sufficient condition for the synchronization manifold to be smooth and persistent under small perturbations. The same examples, however, show that in real applications normal hyperbolicity is *not* sufficient to give quantitative bounds for deformations of the synchronization manifold, i.e., even in the case of normal hyperbolicity two almost identical systems may cause large synchronization errors.

PACS number(s): 05.45.Xt

Synchronization is a phenomenon of interest to fields ranging from celestial mechanics to laser physics, from electronics to communications, and from biophysics to neuroscience [1]. In particular, synchronization in chaotic dynamics [2] has attracted much attention during the last few years because of its role in understanding the basic features of manmade and natural systems. For example, optical communication with chaotic wave forms demonstrated both experimentally [3] and theoretically [4], is only possible because of chaos synchronization between receiver and transmitter. On the other hand, the evidence of chaotic behavior in the brain [5] and the importance of synchronization in perceptive processes of mammals [6] indicate a possible role of chaos synchronization in neural ensembles [7] as well.

Natural language for description of identical and generalized chaos synchronization [8–12] is in terms of invariant, stable, and robust manifolds [13]. In other words, only synchronization phenomena that are described with stable and robust manifolds can be observed in physical experiments. The physical notion of a robust phenomenon contains two separate issues: persistence under small arbitrary perturbations of the system(s) and persistence under small noise. In this paper we address only the question of synchronization that is robust with respect to small perturbations of the dynamical systems involved.

We first repeat the definition of identical and generalized chaos synchronization for arbitrary dynamical systems. Consider a flow $\phi_t(\boldsymbol{z})$ defined on $\mathbb{R}^n$ (or a subset), where the time t may take values from the set of real numbers (in this case the flow is generated by a differential equation) or from the set of integers (in this case the flow is generated by a map). Furthermore, assume that the full system consists of two coupled (sub-) systems with variables (coordinates) $\boldsymbol{x}$ and $\boldsymbol{y}$, with $\boldsymbol{z}=(\boldsymbol{x},\boldsymbol{y})$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are vectors that, in general, may have different dimensions. We say that two coupled systems are in the state of generalized chaos synchronization if there exist a chaotic attractor $\mathcal{A}$ and a function $\boldsymbol{h}:\mathbb{R}^n \to \mathbb{R}^m$, $m<n$, such that $M=\{(\boldsymbol{x},\boldsymbol{y}):\boldsymbol{h}(\boldsymbol{x},\boldsymbol{y})=0\}$ is a stable (i.e., attracting) and smooth invariant manifold, and $\mathcal{A}$ is a subset of M (in other words, the restriction of the dynamics to the invariant manifold is chaotic). In particular, when $M=\{(\boldsymbol{x},\boldsymbol{y}):\boldsymbol{x}=\boldsymbol{y}\}$, identical synchronization (IS) occurs between the two subsystems. This description of generalized synchronization (GS) is compatible with a recently proposed unified definition of synchronization in dynamical systems [14] and it includes phenomena like subharmonic entrainment [11,15].

I. ROBUSTNESS AND NORMAL HYPERBOLICITY

In order to be physically meaningful and experimentally observable any synchronization phenomenon has to be robust, i.e., the synchronization manifold M and its stability properties have to be persistent with respect to (small) arbitrary perturbations of the dynamical systems involved. Therefore, we not only have to ask ''Under what condition is M a stable manifold?'' but also ''Under what condition is M persistent under perturbations?''

In this paper we address the second question that may be answered as follows. There are two linear mutually orthogonal spaces associated with each point $\boldsymbol{z}\in M$: the tangent space T_zM and the normal space N_zM. Let $P:T_zM\times N_zM \to N_zM$ be the orthogonal projection to the normal subspace N_zM. Now we consider the linear part, $D\phi_t(\boldsymbol{z})$, of the flow $\phi_t(\boldsymbol{z})$ at the invariant manifold M. Let $v(t)=D\phi_t(\boldsymbol{z})v(0)$, $v(0)\in T_zM$, $w(t)=PD\phi_t(\boldsymbol{z})w(0)$, $w(0)\in N_zM$.

The invariant manifold M is *stable* if

$$\lim_{t\to\infty}\|w(t)\|=0$$

for all $\boldsymbol{z}\in M$ and all vectors $w(0)\in N_zM$. It is said to be *normally hyperbolic* if

$$\lim_{t\to\infty}\frac{\|w(t)\|}{\|v(t)\|}=0$$

for all $\boldsymbol{z}\in M$ and all nonzero vectors $w\in N_zM$ and $v\in T_zM$. In other words, the rate of normal contraction to the manifold is larger than the tangential one. Normal hyperbolicity is a necessary and sufficient condition for persistence of the invariant manifold under small arbitrary perturbations of the system. The result we have just described is essentially due to Hirsch and Pugh [16–20].

The conditions for stability and normal hyperbolicity can be expressed in terms of Lyapunov exponents. Let ρ be an ergodic invariant measure supported in $\mathcal{A}$. Then there exist m tangential Lyapunov exponents (LE's) (equal to the LE's of $\mathcal{A}$ considered as an attractor of a dynamical system restricted to M) and $n-m$ normal LE's. We write $\lambda_{max}(\rho)$ for the largest normal LE and $\mu_{min}(\rho)$ for the smallest tangential LE. We define $\lambda_{max}=\sup\cup_{\rho\in E}\lambda_{max}(\rho)$, $\lambda_{min}=\inf\cup_{\rho\in E}\lambda_{max}(\rho)$, $\mu_{max}=\sup\cup_{\rho\in E}\mu_{min}(\rho)$, and $\mu_{min}=\inf\cup_{\rho\in E}\mu_{min}(\rho)$, where E is the set of all ergodic invariant probability measures supported in $\mathcal{A}$. The invariant manifold is stable iff

$$\lambda_{max}<0, \tag{1}$$

and the stable manifold is normally hyperbolic if

$$\lambda_{max}<\mu_{min}. \tag{2}$$

If the contraction towards the synchronization manifold is sufficiently strong and if this is true for all trajectories embedded in the chaotic attractor $\mathcal{A}$, the manifold is persistent under perturbations.

We now illustrate the importance of normal hyperbolicity for chaos synchronization using two examples that are both based on the baker map,

$$f_1(x_1,x_2)=\begin{cases}\alpha x_1 & \text{if} \quad x_2<a_1\\ \alpha+\beta x_1 & \text{if} \quad x_2\geqslant a_1\end{cases}$$
$$f_2(x_1,x_2)=\begin{cases}x_2/a_1 & \text{if} \quad x_2<a_1\\ (x_2-a_1)/a_2 & \text{if} \quad x_2\geqslant a_1.\end{cases} \tag{3}$$

As we shall see below, the dynamics restricted to the invariant manifold is governed by $\boldsymbol{f}=(f_1,f_2)$, and therefore we denote the LE's of $\boldsymbol{f}$ with μ (tangential LE's). For the baker map, a_1, a_2, α, and β are positive real numbers (parameters) such that $\alpha\leqslant\beta$, $\alpha+\beta=1$, and $a_1+a_2=1$. The chaotic attractor of the baker map has a natural invariant measure that is uniform in x_2 and varies wildly in the x_1 direction. It is easy to see that the LE's of this attractor with respect to the natural measure are $\mu_1(\rho_{nat})=a_1\ln(1/a_1)+a_2\ln(1/a_2)>0$ and $\mu_2(\rho_{nat})=a_1\ln\alpha+a_2\ln\beta<0$ [21]. In addition one can show that for all ergodic measures ρ, $\ln\alpha\leqslant\mu_2(\rho)\leqslant\ln\beta$. Hence, the smallest negative LE of the baker map only is given by $\mu_{min}=\ln\alpha$, and the largest negative LE by $\mu_{max}=\ln\beta$. We stress that here we consider the baker map because it offers the possibility of graphical illustrations of the synchronization manifold and some of the results can be obtained analytically. However, the main results of this paper are not restriced to the baker map but hold for any other dynamical system. For example, similar results for the Lorenz system are reported in Ref. [19].

II. NORMAL HYPERBOLICITY AND GS

We now show, with an example, that the absence of normal hyperbolicity leads to a loss of smoothness for GS manifolds. To do this, we consider the baker map that drives a linear system,

$$\boldsymbol{x}(n+1)=\boldsymbol{f}(\boldsymbol{x}(n)),$$
$$y(n+1)=ay(n)+\varepsilon\cos[2\pi x_1(n)], \tag{4}$$

where $\varepsilon>0$ is the coupling parameter. For $\varepsilon=0$ and $|a|<1$ the full system possesses an attracting invariant manifold $M=\{(\boldsymbol{x},y):y=0\}$. If the coupling is switched on ($\varepsilon>0$) the full system and this manifold are perturbed. For this example the new (perturbed) manifold can be computed analytically with the state of the response y being given by an explicit function

$$y=\varepsilon\sum_{j=1}^{\infty}a^{j-1}\cos[2\pi f_1^{-j}(x_1,x_2)]=\varepsilon\sum_{j=1}^{\infty}g_j(\boldsymbol{x}) \tag{5}$$

of the state variables x_1 and x_2 of the drive [20]. Since for $|a|<1$ the functions g_j are bounded and continuous, the sum converges uniformly and function (5) is therefore properly defined, continuous, and its graph is a globally attracting invariant manifold of Eq. (4). For the baker map (3) the inverse f_1^{-j} depends on x_1 only and therefore the function (5) and the resulting graph are independent of x_2 in this case. Furthermore, the size of the perturbation (i.e., the value of ε) is in this case arbitrary, because the response system is linear and ε can without loss of generality be chosen to be unit. The LE describing the contraction normal to this manifold equals $\ln a$ and the tangential LE's are those of the baker map. We now want to address the question of whether function (5) and the corresponding manifold are differentiable or not. From the numerical experiment of Hunt *et al.* [10] we know that the function (5) is differentiable for $a<\alpha$, and it is not differentiable for $a>\beta$. We now give a heuristical explanation for this behavior.

From Eq. (3) it follows that Eq. (5) can be rewritten as

$$y=\varepsilon\sum_{j=1}^{\infty}a^{j-1}\cos(2\pi[\alpha^{-j_1}\beta^{-j_2}x_1+\Phi_j]), \tag{6}$$

where $j_1+j_2=j$ and Φ_j is a real number. We first consider the case when $\alpha=\beta$. Then Eq. (6) becomes

$$y=\frac{\varepsilon}{a}\sum_{j=1}^{\infty}a^j\cos(2\pi b^{-j}x_1+\Phi), \tag{7}$$

where $b=\alpha=\beta$. Equation (7) is the famous Weierstrass function, which is known to be nowhere differentiable for $b<a$ [22]. This is essentially due to the fact that the infinite sum,

$$\frac{\partial y}{\partial x_1}\equiv-\varepsilon\frac{2\pi}{a}\sum_{j=1}^{\infty}\left(\frac{a}{b}\right)^j\sin(2\pi b^{-j}x_1+\Phi) \tag{8}$$

is divergent for all x_1 if $b<a$. Therefore, for $\alpha=\beta$, we have a sharp transition from a differentiable manifold ($a<\alpha=\beta$) to a nowhere differentiable manifold ($a>\alpha=\beta$) at the critical point ($a=\alpha=\beta$).

However, in general $\alpha\neq\beta$ and we shall assume in the following $\alpha<\beta$. Then, there exist two problems concerning a rigorous treatment of Eq. (6). The first one is that Eq. (6) can be rewritten as Eq. (7) only approximatively. If the driving trajectory is periodic, then the term $\alpha^{-j_1}\beta^{-j_2}$ is for large j proportional to b^{-j} with $b=[\alpha^{k_1}\beta^{k_2}]^{-k}$, where $k=k_1+k_2$ is the period of the driving trajectory, k_1 is the number of points with slope α, and k_2 is the number of points with slope β. Thus for example, for $k_1=k_2=1$, $\alpha^{-j_1}\beta^{-j_2}$ may alternate between two values $\alpha^{-1}b^{-j}$ and b^{-j}, where $b=\alpha\beta$. For chaotic driving trajectories, the term $\alpha^{-j_1}\beta^{-j_2}$ is for large j proportional to the Lyapunov number of the driving trajectory, that is, to $b^{-j}=[\alpha^{a_1}\beta^{a_2}]^{-j}$ (this follows from the existence of Lyapunov exponents of the baker map).

The second problem results from the fact that periodic points of the baker map (drive system) are dense. This means that in a neighborhood of any point there exists an infinite number of points with different values of the asymptotic constant b. In particular, for $\alpha<a<\beta$ we can always find close to a point with $a<b$ another one with $a>b$. This raises the question of what we mean exactly by differentiability (or smoothness) of a function $\phi:\Lambda\to[0,1]$ for an arbitrary set $\Lambda\subset[0,1]$, which is beyond the scope of this paper.

The above discussion shows that in general $b\in[\alpha,\beta]$. Thus, for $a<\alpha$ it follows that $a<b$ for all trajectories of the baker map and the synchronization manifold is a C^1 function for all x_1. Note that $a<\alpha$ coincides with condition (2) for normal hyperbolicity. In the case $a>\beta$, condition (2) for normal hyperbolicity is violated everywhere on the manifold which is then likely to be the graph of a continuous but nowhere differentiable function. We conjecture that this is indeed the case, because from $\beta<a$ it follows that $b<a$ for all trajectories of the baker map and this suggests that Eq. (6) is a nowhere differentiable function. What happens for $\alpha<a<\beta$? The answer to this question is left to future investigations, although our numerical experiments indicate that as a increases, the invariant manifold looses smoothness first at individual periodic orbits (for $\alpha\leqslant a\leqslant\alpha^{a_1}\beta^{a_2}$). Then, for $\alpha^{a_1}\beta^{a_2}\leqslant a\leqslant\beta$, the invariant manifold looses smoothness at chaotic orbits and finally for $a>\beta$ the invariant manifold is nowhere differentiable function for all x_1. We conjecture that the transition from a C^1 to a C^0 manifold is typically not abrupt, but that the invariant manifold locally loses its smoothness when a increases from α to β. Note that one can generalize this concept to C^k manifolds and in this case Eq. (2) has to be replaced by a similar relation. The coupled system (4), for example, possesses C^k synchronization manifolds for $a<\alpha^k$. We also note that since $\alpha<1$ the synchronization manifold for our example is never C^∞.

III. NORMAL HYPERBOLICITY AND IS

In the literature on identical chaos synchronization, stability of the invariant manifold is the only requirement for synchronization. This is due to the fact that for identical synchronization, the synchronization manifold, $\boldsymbol{x}=\boldsymbol{y}$, is smooth even when it is not stable. We show now, by means of an example, that slight mismatch of the parameters in the case $\lambda_{max}>\mu_{min}$ may cause the invariant manifold to lose its smoothness and become a fractal set. For our example,

$$\boldsymbol{x}(n+1)=\boldsymbol{f}(\boldsymbol{x}(n)),\qquad(9)$$

$$\boldsymbol{y}(n+1)=\tilde{\boldsymbol{f}}[\boldsymbol{y}(n)]+\mathbf{h}(x(n),\boldsymbol{y}(n)),$$

we use $\boldsymbol{f}$ and $\tilde{\boldsymbol{f}}$ to be the baker map (3) with different parameters, and the feedback linearization coupling

$$\mathbf{h}(\boldsymbol{x},\boldsymbol{y})=\tilde{\boldsymbol{f}}(\boldsymbol{x})-\tilde{\boldsymbol{f}}(\boldsymbol{y})+A(\boldsymbol{y}-\boldsymbol{x}),$$

with a stable 2×2 matrix A. Using this active passive decomposition [23] of the drive system we obtain a response system that synchronizes globally with the drive. We further assume that $a_{12}=0$ so that the eigenvalues of A are its diagonal elements. We also assume that $0<a_{11}\leqslant a_{22}<1$. If we define $\boldsymbol{z}=\boldsymbol{y}-\boldsymbol{x}$, then the dynamics of Eq. (9) can be rewritten as

$$\begin{aligned}\boldsymbol{x}(n+1)&=\boldsymbol{f}(\boldsymbol{x}(n)),\\ \boldsymbol{z}(n+1)&=A\boldsymbol{z}+\tilde{\boldsymbol{g}}(\boldsymbol{x}(n)),\end{aligned}\qquad(10)$$

where $\tilde{\boldsymbol{g}}=\tilde{\boldsymbol{f}}-\boldsymbol{f}$. Since $\tilde{\boldsymbol{g}}$ is a bounded function, we write $\tilde{\boldsymbol{g}}=\varepsilon\boldsymbol{g}$, where $\varepsilon=\max\tilde{\boldsymbol{g}}$. Normal LE's are $\ln a_{11}$ and $\ln a_{22}$, while tangential LE's are those of the baker map. Condition (2) can be rewritten as $a_{22}<\alpha$.

In the ideal case, when the parameters of drive and response are the same the synchronization manifold $\boldsymbol{z}=0$ (or $\boldsymbol{x}=\boldsymbol{y}$) is globally stable. We shall consider now a perturbation of the full coupled systems and its synchronization manifold in terms of parameter mismatch between both baker maps. By resubstitution of Eq. (10), one can show that the GS manifold is in this case given by the function

$$\boldsymbol{z}=\varepsilon B^{-1}\sum_{j=1}^{\infty}C^{j-1}B\boldsymbol{g}(\boldsymbol{f}^{-j}(\boldsymbol{x})),\qquad(11)$$

where B and $C=\mathrm{diag}\,(a_{11},a_{22})$ are 2×2 matrices with $b_{11}=b_{22}=1$, $b_{12}=0$, and $b_{21}=a_{21}/(a_{22}-a_{11})$. Following similar arguments as above we may conclude that for $a_{22}<\alpha$ Eq. (11) describes a smooth manifold and we observe robust synchronization [see Fig. 1(a)]. On the other hand, for $a_{11}>\beta$ the manifold (11) has a fractal structure, as can be seen in the Fig. 1(b).

We now ask the question, ''How small should the perturbation (mismatch of the parameters) be so that the perturbed manifold is close to the original (unperturbed)?'' Normal hyperbolicity guarantees only persistence and smoothness, but says nothing about the actual deformation of the manifold due to a perturbation. Using example (10) we shall demonstrate that even for fixed, small normal LE's this deformation may become arbitrary large depending on the coupling. The unperturbed manifold is $\boldsymbol{z}=0$. In order to study the deformation of this manifold we estimate $\max|\boldsymbol{z}|$ for $a_{21}\gg1$ to be $\max|\boldsymbol{z}|\sim\varepsilon a_{21}$. The parameter ε is a measure of the parameter mismatch or perturbation and we assume that it is small (but not arbitrary small, because we want to apply this reasoning also to experimental systems). If the parameter

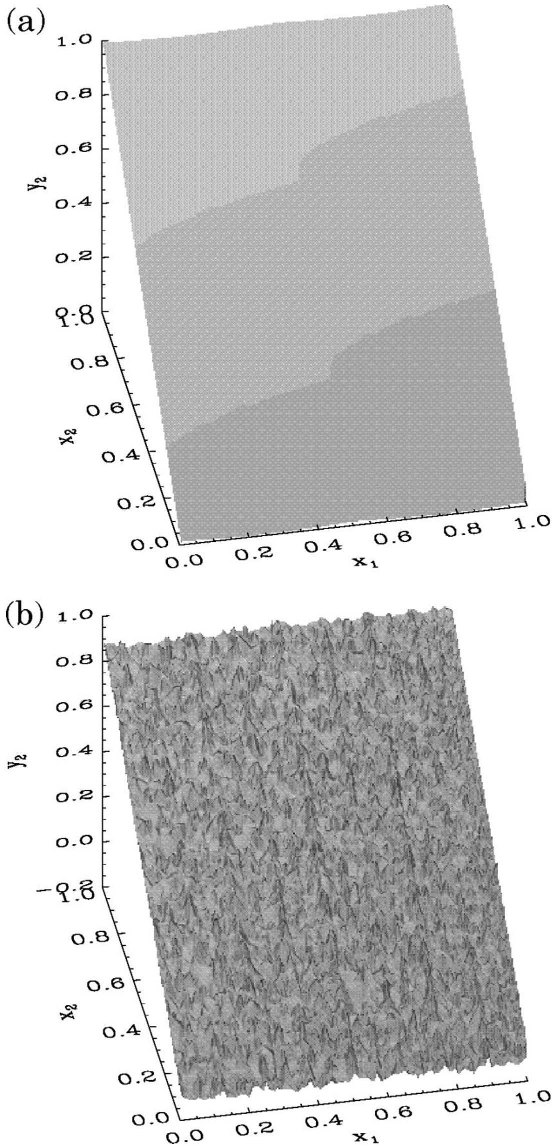

FIG. 1. Graph y_2 vs (x_1, x_2) of the synchronization manifold of the system (9) of two coupled baker maps (3). Parameter values are $\alpha=0.5$, $\beta=0.5$, $a_1=0.45$, $a_2=0.55$, $\tilde{\alpha}=0.49$, $\tilde{\beta}=0.51$, and $a_{21}=1$. (a) Smooth manifold for $a_{11}=0.1$ and $a_{22}=0.12$. (b) Fractal manifold for $a_{11}=0.8$ and $a_{22}=0.82$.

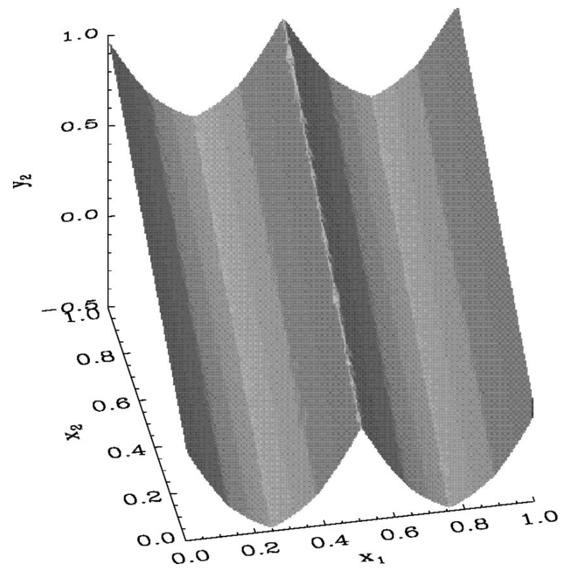

FIG. 2. Graph y_2 vs (x_1, x_2) of the synchronization manifold of the system (9) of two coupled baker maps (3). Parameter values are $\alpha=0.5$, $\beta=0.5$, $a_1=0.45$, $a_2=0.55$, $\tilde{\alpha}=0.49$, $\tilde{\beta}=0.51$, $a_{11}=0.1$, $a_{22}=0.12$, and $a_{21}=50$. The manifold is smooth due to normal hyperbolicity but deviates strongly from the identical synchronization manifold $\mathbf{x}=\mathbf{y}$ [compare Fig. 1(a)].

a_{21} becomes large, the GS manifold (11) deviates significantly from the IS manifold and we observe large synchronization errors (for our system this error grows linearly with a_{21}) although a_{21} *does not* influence the normal contraction rates (and LE's) ! Therefore, even in the case when the synchronization manifold is normally hyperbolic, the systems may possess a parameter that influences very strongly the synchronization (manifold). To demonstrate this effect, we show in Fig. 2 the same graph as in Fig. 1(a) but now for $a_{21}=50$. As can be seen, the manifold remains smooth (as we expect it, because condition (2) is fulfilled) but deviates significantly from the IS manifold $M=\{(\boldsymbol{x},\boldsymbol{y}):\boldsymbol{x}=\boldsymbol{y}\}$. Note that in this case $b_{21}=2500$ and the eigenvectors $[1,-b_{21}]$ and $[0,1]$ of the matrix A are almost parallel and all perturbations orthogonal to these eigenvectors are strongly amplified before they converge to zero. Due to the parameter mismatch such large deviations from $\mathbf{z}=0$ are excited again and again and this mechanism is the deeper reason for the strong sensitivity of the synchronization manifold on perturbations of the coupled systems.

In this paper we have addressed the problem of robustness of synchronization (manifolds) with respect to (small) perturbations of the underlying dynamical systems. It turns out that two different aspects have to be distinguished: (i) persistence of qualititive features such as the existence or smoothness of the manifold and (ii) boundedness of (quantitative) deformations of the manifold. Both robustness features have been found to be independent from each other. Robustness in the first sense is guaranteed for normally hyperbolic systems and depends on normal contraction rates that can be measured in terms of normal and tangential Lyapunov exponents. The (non-) boundedness of deformations of the manifold does not depend on the contraction rates but is related to contraction (and expansion) directions. Since for many physical systems and applications not only smoothness of the synchronization manifold but also quantitative deviations and differences are of importance normal hyperbolicity seems in this sense not to be a sufficient condition for robust synchronization.

ACKNOWLEDGMENTS

We thank N. Rulkov, L. Pecora, and L. Junge for stimulating discussions on chaos synchronization. This work was supported in part by the ARO (Grant No. DAAG55-98-1-0269), by the DOE (Grant No. DE-FG03-95ER14516), by the NSF (Grant No. NCR-9612250 and CAREER Grant No. PHY-972236), by the Deutsche Forschungsgemeinschaft (Grant No. Pa 643/1-1), and by NATO (Grant No. PST.CLG 974981).

[1] Chaos **7** (1997); IEEE Trans. Circuits Syst., part I **44** (1997), and references therein.
[2] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **69**, 32 (1983); V. S. Afraimovich, N. N. Verichev, and M. I. Rabinovich, Radiophys. Quantum Electron. **29**, 747 (1986); L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[3] G. D. VanWiggeren and R. Roy, Science **279**, 1198 (1998)); G. D. VanWiggeren and R. Roy, Phys. Rev. Lett. **81**, 3547 (1998).
[4] H. D. I. Abarbanel and M. Kennel, Phys. Rev. Lett. **80**, 3153 (1998).
[5] C. Skarda and W. J. Freeman, Behav. Brain Sci. **10**, 161 (1987).
[6] W. Singer, Annu. Rev. Physiol. **55**, 349 (1993).
[7] S. J. Schiff, P. So, T. Chang, R. E. Burke, and T. Sauer, Phys. Rev. E **54**, 6708 (1996).
[8] N. F. Rulkov, M. M. Sushchik, L. S. Tsimring, and H. D. I. Abarbanel, Phys. Rev. E **51**, 980 (1995).
[9] L. Kocarev and U. Parlitz, Phys. Rev. Lett. **76**, 1816 (1996).
[10] B. R. Hunt, E. Ott, and J. A. Yorke, Phys. Rev. E **55**, 4029 (1997).
[11] U. Parlitz, L. Junge, and L. Kocarev, Phys. Rev. Lett. **79**, 3158 (1997).
[12] R. Brown, Phys. Rev. Lett. **81**, 4835 (1998).
[13] K. Josic, Phys. Rev. Lett. **80**, 3053 (1998).
[14] R. Brown and L. Kocarev, Chaos (to be published).
[15] N. Rulkov and M. Sushchik, Phys. Lett. A **214**, 145 (1996).
[16] F.C. Hoppensteadt and E.M. Izhikevich, *Weakly Connected Neural Networks*, Appl. Math. Sci. Vol. 126 (Springer, New York, 1997).
[17] M. Hirsch and C. Pugh, in *Global Analysis*, AMS Proceedings of the Symposium on Pure Mathematics, (American Math. Soc., Providence, RI, 1970), Vol. 14, p. 133.
[18] S. Wiggins, *Normally Hyperbolic Invariant Manifolds in Dynamical Systems* (Springer-Verlag, New York, 1994).
[19] L. Kocarev, J. Bröcker, and U. Parlitz, in *Proceedings of the 1999 International Symposium on Nonlinear Theory and Its Applications* (NOLTA '99), Waikoloa, Hawaii, 1999 (IEICE, Tokyo, 1999), pp. 37–40.
[20] J. Stark, Physica D **109**, 163 (1997).
[21] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, 1993).
[22] G. H. Hardy, Trans. Am. Math. Soc. **17**, 301 (1916).
[23] L. Kocarev and U. Parlitz, Phys. Rev. Lett. **74**, 5028 (1995); U. Parlitz, L. Kocarev, T. Stojanovski, and H. Preckel, Phys. Rev. E **53**, 4351 (1996).